Schur's Lemma

Prop (Schur's Lemma):

Let $\varphi : G \to GL(V)$, $\psi : G \to GL(W)$ irreducible

and $T \in Hom_G(\varphi, \psi)$ a morphism

Then either $T = 0$ or T is an isomorphism

(1) If $\varphi \not\sim \psi$, then $Hom_G(\varphi, \psi) = \{0\}$

(2) If $\varphi \sim \psi$, then $\dim Hom_G(\varphi, \psi) = 1$

In particular, $Hom_G(\varphi, \varphi) = \{\lambda I, \lambda \in \mathbb{C}\}$

Proof: $T \in Hom_G(\varphi, \psi)$, φ, ψ irreducible.

$\ker(T) \leq V$, $T(V) \leq W$ invariant subspaces

If $T \neq 0$, then $\ker(T) \neq V$, $T(V) \neq \{0\}$

$\Rightarrow$ $\ker(T) = \{0\}$ $T(V) = W$

$\Rightarrow$ T is isomorphism!

(1) $\varphi \not\sim \psi$, irreducible.

$T \in \mathrm{Hom}_G(\varphi, \psi) \Rightarrow T = 0$

(2) $\varphi \sim \psi$ irreducible

<u>special case:</u> $\varphi = \psi$, $T \in \mathrm{Hom}_G(\varphi, \varphi)$

so $T : V \to V$, has an eigenvalue $\lambda \in \mathbb{C}$

so $\underset{\mathrm{Hom}_G(\varphi,\varphi)}{T - \lambda I \ \cap}$ <u>not an isomorphism</u> of V

$\Rightarrow T - \lambda I = 0 \Rightarrow T = \lambda I$

so $\mathrm{Hom}_G(\varphi, \varphi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$

If $\varphi \sim \psi$, irreducible.

Pick $S \in \mathrm{Hom}_G(\psi, \varphi)$, equivalence, S^{-1} exists

If $\underbrace{T \in \mathrm{Hom}_G(\varphi, \psi)}$, the $\underset{\lambda I}{\overset{}{TS}} \in \mathrm{Hom}_G(\psi, \psi)$

$\Rightarrow T = \underline{\lambda S^{-1}}$ so $\underset{\text{1-dim'l}}{\mathrm{Hom}_G(\varphi, \psi) = \{\lambda S^{-1}\}}$